8

SINGULARITY STRUCTURE

It has been seen that singularities inevitably occur in the solutions describing the interaction region of colliding plane waves. Using the line element (6.20), we have in this region

$$e^{-U} = f(u) + g(v) \tag{8.1}$$

where $f(u)$ and $g(v)$ are monotonically decreasing functions for positive arguments. It is therefore inevitable that some kind of singularity will occur on the hypersurface $f + g = 0$. It can also be seen from (6.17), that this is the hypersurface on which the two opposing waves mutually focus each other, as the contraction of each wave here becomes unbounded. This was anticipated by the discussion in Section 5.3. It is now appropriate to consider whether the caustics formed in this way correspond to mere coordinate singularities, or whether they are necessarily curvature singularities as they are in the Khan–Penrose solution.

It has also been pointed out previously that coordinate singularities necessarily occur in the regions II and III that contain the approaching waves. The character and significance of these singularities must also be considered in this chapter.

8.1 Singularities

According to the general theory of relativity, space-time is represented by a connected C^∞ Hausdorff manifold M together with a locally Lorentz metric g. A singularity[1] in the space-time is indicated by incomplete geodesics or incomplete curves of bounded acceleration (Hawking and Ellis, 1973).

By definition, space-time is smooth and does not contain any irregular points. It follows that a singularity may normally be considered as occurring only at a boundary of space-time. Unfortunately, no single definition of a singularity has yet been found which is applicable to all situations.

One of the more useful ways of attaching a boundary to a singular space-time is by a b(bundle)-boundary construction. The b-boundary is the projection into a space-time of a natural boundary attached to a higher-dimensional Riemannian manifold. In the standard b-boundary construction, the Riemannian manifold is the bundle of frames over space-time having a positive definite metric induced by the affine connection. Boundary points of the frame bundle are determined by giving end points to all Cauchy sequences which do not converge in the frame bundle. The bundle boundary is then projected down to make a boundary for the space-time.

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[1] This brief review follows closely that of Konkowski and Helliwell, 1989. For a more complete review see Tipler, Clarke and Ellis (1980)

According to the classification scheme devised by Ellis and Schmidt (1977), singularities in maximal, four-dimensional space-times can be divided into three basic types: quasiregular, non-scalar curvature and scalar curvature. This scheme describes the singularity structure of a space-time (M, g) on which the Riemann tensor is C^k. It uses a b-boundary construction to determine the location of singular points.

If the b-boundary is non-empty, there are only two possibilities. Either a point q in the b-boundary may be a C^r $(r \geq 0^-)$ regular boundary point if the space-time (M, g) can be embedded in a larger space-time (M', g') such that the Riemann tensor is C^r and q is an interior point in M', or it may be a C^r singular boundary point if the space-time (M, g) is not extendable through q in a C^r way.

A singular boundary point q can then be classified according to this scheme. It may be a C^k (or C^{k-}) *quasiregular singularity* if all components of the Riemann tensor and its first k derivatives evaluated in an orthonormal frame parallel propagated along an incomplete geodesic ending at q are C^0 (or C^{0-}). Such frames are called PPON frames. It may alternatively be a C^k (or C^{k-}) curvature singularity if this is not true. In this case it may either be categorized as a C^k (or C^{k-}) *non-scalar curvature singularity* if all scalars in the metric tensor $g_{\mu\nu}$, the alternating symbol $\epsilon_{\kappa\lambda\mu\nu}$, the Riemann tensor and its first k derivatives are bounded, that is, tend to a C^0 (or C^{0-}) function. Alternatively, it may be a C^k (or C^{k-}) *scalar curvature singularity* or a *scalar polynomial curvature singularity* if some scalar does not tend to a C^0 (or C^{0-}) function.

The most familiar class of singularities are the scalar curvature singularities. These include the 'big bang' and 'black hole' types of singularity which closely correspond to one's intuitive concept of a real physical singularity. As such a singularity is approached, some physical quantities diverge and all observers feel unbounded tidal forces.

The non-scalar curvature and quasiregular singularities are much less well understood and have been less fully investigated. Consider, for example, a space-time with a non-scalar curvature singularity. No curvature scalars diverge in this case, yet some components of the Riemann tensor evaluated in a PPON frame along an incomplete curve do not tend to finite limits as the singularity is approached. The physical effect of this is that all test particles which fall into the non-scalar curvature singularity feel infinite tidal forces, but observers can move arbitrarily close to the singularity on other curves and feel no untoward effects.

Finally, consider a space-time with a quasiregular singularity. In all reasonable frames the Riemann tensor is completely finite. In this case, observers near a quasiregular singularity, including those who fall into the singularity itself, do not feel unbounded tidal forces.

All three types of singularity are found in colliding plane wave solutions. In the following sections and chapters the singularity and global structure of particular solutions will be analysed in more detail.

8.2 The singularity in region IV

In the exact solution of Khan and Penrose (1971) described in Chapter 3, there is a scalar curvature singularity in region IV on the hypersurface $f+g=0$. However, it is not clear whether or not this type of singularity will occur in other solutions, particularly as the Khan–Penrose solution contains impulsive gravitational waves.

This question has been considered by Szekeres (1972), who found that, for colliding gravitational waves with aligned linear polarization, a curvature singularity is inevitable. This work was generalized by Sbytov (1976) to plane gravitational waves with arbitrary polarization, with the same result. However, more recently, counterexamples to these conclusions have been obtained, in the non-aligned case by Chandrasekhar and Xanthopoulos (1986*c*), and in the aligned case by a degenerate solution of Ferrari and Ibañez (1987*b*) and by the algebraically general solutions of Feinstein and Ibañez (1989). In these cases, the curvature scalars remain bounded on the hypersurface $f+g=0$, and scalar polynomial curvature singularities occur in the extensions of the solution through the focusing singularity. These exceptional cases will be considered in detail later on. For the present, it may simply be observed that Szekeres and Sbytov had omitted to include these cases.

Another counterexample proposed by Stoyanov (1979) has proved to be incorrect as it does not satisfy the required boundary conditions (see Section 10.2).

The same question applied to colliding electromagnetic waves is more complicated, and the answers are less conclusive. Generally, one may expect that, for a combination of gravitational and electromagnetic waves, a curvature singularity in region IV will usually occur. However, for purely electromagnetic waves it is less clear. In the first exact solution of this type, given by Bell and Szekeres (1974), the singularity on the hypersurface $f+g=0$ was shown to be only a coordinate singularity that could easily be removed by a coordinate transformation. This solution, which is conformally flat in the interaction region, will be described in Chapter 14 together with the full analysis of its singularity structure as given by Clarke and Hayward (1989). Other type D electrovac solutions have been obtained by Chandrasekhar and Xanthopoulos (1987*a*) and Papacostas and Xanthopoulos (1988). Further algebraically general solutions without curvature singularities can easily be constructed as will be indicated in Section 17.2. All these have quasiregular singularities that are interpreted as Cauchy horizons on the surface $f+g=0$. These will be described later.

We may conclude that, in all cases, the opposing waves mutually focus each other onto the hypersurface $f+g=0$, on which the contraction of the waves is unbounded and the line element (6.20) is singular. Usually this will be a scalar polynomial curvature singularity, but a large class of of significant exceptions occurs.

It is convenient to point out at this stage that, for colliding gravitational and electromagnetic waves, two of the scalar polynomial invariants (Penrose and Rindler, 1986) are given by

$$I = 2\Psi_0\Psi_4 + 6\Psi_2^2, \qquad J = 6(\Psi_0\Psi_4 - \Psi_2^2)\Psi_2. \tag{8.2}$$

It follows from this that, in order to prove the existence of a scalar polynomial curvature singularity, it is sufficient merely to show that the component Ψ_2 is unbounded. In this case either I or J must be unbounded, and a curvature singularity occurs.

It would, of course, be useful to have a number of general theorems that could be used to determine the singularity structure of particular classes of solutions. Apart from the earlier results of Szekeres (1972) and Sbytov (1976), the only general theorem to date is that of Tipler (1980). This is in fact a straightforward generalization of a theorem of Penrose (1965*b*) (see also Hawking and Ellis 1973, p.263). Quoting it directly:

> **Theorem 8.1** (Tipler) *Let (M,g) be a space-time with g at least C^2, and suppose (M,g) has two globally defined commuting space-like Killing vector fields ∂_x and ∂_y, which together generate plane symmetry. If, (1) the null convergence condition holds; (2) at least one of the six quantities Ψ_0, Ψ_4, Φ_{00}, Φ_{22}, σ, λ is non-zero at some point p in (M,g); and (3) through the point p there is a space-like partial Cauchy surface S, which is everywhere tangent to ∂_x and ∂_y, and S is non-compact in the space-like direction normal to ∂_x and ∂_y; then (M,g) is null incomplete.*

This is an interesting theorem which seems to prove the existence of singularities for a large class of colliding plane waves. However, it requires that the metric be at least C^2 everywhere. It thus excludes situations involving impulsive gravitational waves. It does not apply therefore to the solutions of Khan and Penrose (1971), Ferrari and Ibañez (1987*b*), and the exceptional solution of Chandrasekhar and Xanthopoulos (1986*b*). Nor does it apply to the colliding electromagnetic wave solution of Bell and Szekeres (1974) in which impulsive gravitational waves are generated by the collision. In addition, it proves only geodesic incompleteness and says nothing about curvature singularities.

There exists, however, a very large class of exceptional solutions in which curvature singularities do not occur. For gravitational waves these include a degenerate Ferrari–Ibañez (1987*b*) solution and the solution of Chandrasekhar and Xanthopoulos (1986*b*), which are respectively parts of the Schwarzschild and Kerr space-times. These have been described in more detail by Ferrari and Ibañez (1988), and will be described in Sections 10.5 and 13.3. They contain impulsive wave components, and therefore do not satisfy the conditions of Tipler's theorem. In all these solutions, the singularity that occurs when $f + g = 0$ corresponds to a Cauchy horizon. This may be followed by either a space-like, or a time-like curvature singularity. The latter case would indicate that, if it were possible for real observers to pass through the horizon, then most would miss this singularity.

A further class of exceptional vacuum solutions in which the curvature singularity is replaced by a Cauchy horizon has been obtained by Feinstein and Ibañez (1989). These solutions contain a subclass in which the approaching waves have smooth wave fronts and the metric is everywhere at least C^2. They thus satisfy the conditions of Tipler's theorem. They are geodesically incomplete, but the focusing hypersurface in this case is only a quasiregular singularity. These solutions thus illustrate the fact that a proof of null incompleteness does not necessarily imply the existence of a scalar polynomial curvature singularity.

The Killing–Cauchy horizons that occur in these solutions have particular significance as the caustics formed by the mutual focussing of the opposing waves. They have also been further investigated by Yurtsever (1987), who has shown that they are unstable against plane-symmetric perturbations. It is therefore reasonable to conclude that the existence of space-like singularities is likely to be a generic feature of colliding plane wave solutions. This conclusion is also supported by the work of Chandrasekhar and Xanthopoulos (1987*b*), who have shown that the presence of an arbitrarily small amount of dust will change a horizon into a curvature singularity.

In these exceptional cases, it is necessary for the approaching waves to take very specific forms in order to achieve the appropriate solution in region IV. The instability of the horizons in these solutions can easily be demonstrated in that they require very specific forms for the initial functions $f(u)$ and $g(v)$. Any slight variation in these functions would change the Cauchy horizon into a curvature singularity.

Most of the exceptional solutions that have been explicitly obtained are of algebraic type D, although the solutions of Feinstein and Ibañez are algebraically general. Perhaps it should also be remarked that algebraic type D solutions do not necessarily have horizons rather than singularities, a counterexample being one of the degenerate Ferrari–Ibañez (1987*b*) solutions.

An alternative proof of Tipler's Theorem 8.1 has been given by Yurtsever (1988*b*). This emphasizes the role and necessity of the assumption of a strict plane symmetry. In this approach, the exceptional solutions that have Killing–Cauchy horizons in the interaction region do not satisfy the condition of strict plane symmetry as defined by Yurtsever.

The structure of the singularity in the interaction region has been further analysed by Yurtsever (1988*c*, 1989*a*), who has shown that the metric is asymptotic to an inhomogeneous Kasner solution as the singularity is approached. Initially (Yurtsever 1988*c*), he considered the case when the approaching waves have constant aligned polarization and obtained explicit expressions which relate the asymptotic Kasner exponents along the singularity to the initial data posed along the wave fronts of the approaching waves. From these expressions it is clear that, for specific choices of initial data, the curvature singularity formed by the interacting waves degenerates to a coordinate singularity. It can also be concluded that these Killing–Cauchy horizons are unstable against small but generic perturbations of the initial data and that, in a very precise sense, 'generic' initial data always produce all-embracing space-like curvature singularities. In the subsequent paper (Yurtsever, 1989*a*), he has shown that these same conclusions are also reached in the case when the polarization of the approaching waves is arbitrary.

8.3 The Khan–Penrose solution

The solution of Khan and Penrose (1971) which has already been discussed in detail in Chapter 3, may now be reconsidered. It was shown there in particular, that the scalar invariants are unbounded on the hypersurface $f+g=0$ in region IV, where $f=\frac{1}{2}-u^2$ and $g=\frac{1}{2}-v^2$. This clearly demonstrates the existence of a scalar curvature singularity on this hypersurface. In addition to this, there are apparent coordinate singularities extending from it into regions II and III. The naïve singularity structure of this solution is thus as represented in Figure 3.2.

It is now necessary to consider in more detail the character of the apparent singularities in these initial regions II and III. As described in Section 3.3, if it is possible for real particles to pass through them, then it will be possible for those particles to subsequently look back and observe the naked singularity in region IV. If correct, this would provide a counterexample to the cosmic censorship hypothesis. It is more likely, however, that these singularities will prove to be more than artificial coordinate singularities that particles can pass through.

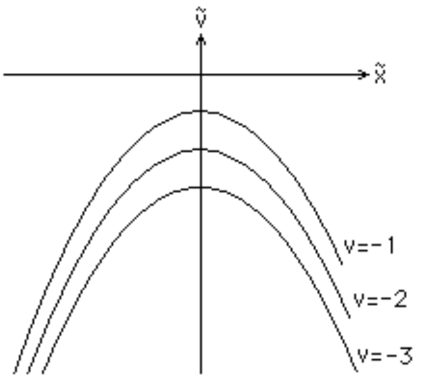

Figure 8.1 Sections through surfaces v = constant, $y = 0$, when $u = 0$, for three different (negative) values of v.

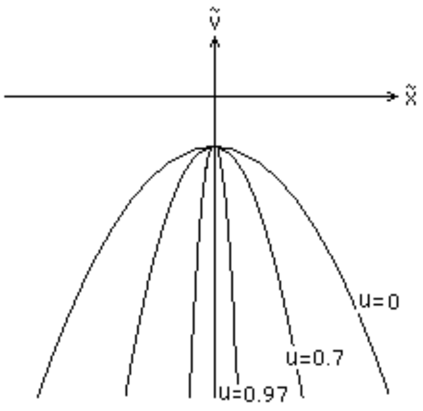

Figure 8.2 Sections through the surface v = constant, $y = 0$, for three different values of u.

This question has been discussed in an interesting paper by Matzner and Tipler (1984). Because of the obvious symmetry, we may concentrate on the singularity in region II. The first point to note is that, since the initial waves are impulsive, the interiors of both regions I and II are flat. For $v \leq 0$, curvature only occurs on the null boundary $u = 0$. The line elements for regions I and II are respectively

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \tag{8.3}$$

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-u)^2\mathrm{d}x^2 - (1+u)^2\mathrm{d}y^2. \tag{8.4}$$

That these both describe flat space-time can be demonstrated by transforming (8.4) to the form (8.3) by putting

$$u = \tilde{u}, \qquad v = \tilde{v} + \frac{\frac{1}{2}\tilde{x}^2}{1-\tilde{u}} - \frac{\frac{1}{2}\tilde{y}^2}{1+\tilde{u}}, \qquad x = \frac{\tilde{x}}{1-\tilde{u}}, \qquad y = \frac{\tilde{y}}{1+\tilde{u}}. \tag{8.5}$$

It may be observed that the coordinate singularity $u = 1$ is effectively removed by this transformation, and there appears no *a priori* reason preventing the continuation of the coordinates $\tilde{x}^\mu$ through it.

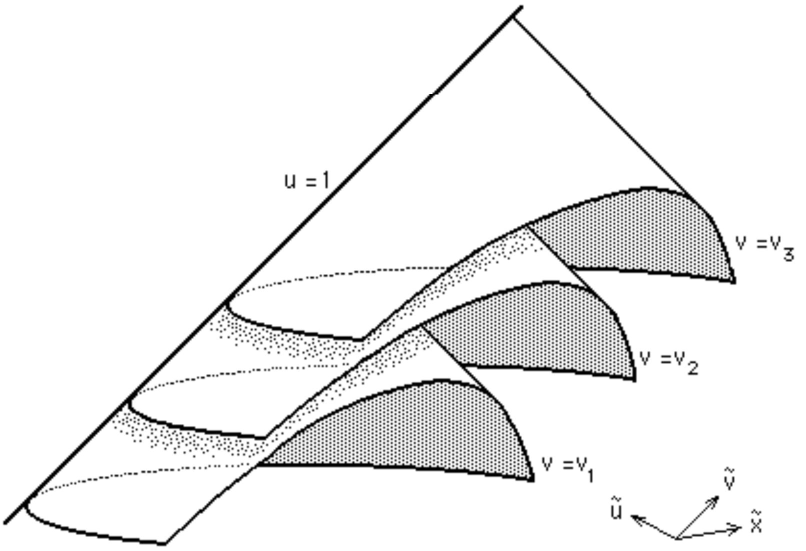

Figure 8.3 Surfaces $v =$ constant, $y = 0$, $0 < u < 1$, for different (negative) values of v. Notice that the line $u = 1$ is common to all surfaces.

Matzner and Tipler proceed to investigate the properties of the null hypersurfaces $v =$ constant, working with the null Minkowski coordinates $\tilde{x}^\mu$. Projections of these hypersurfaces are illustrated in Figures 8.1, 8.2 and 8.3. It can be seen that as $u \to 1$ the curvature of these hypersurfaces diverges. It should also be noticed that the surfaces $v =$ constant, considered as embedded in the three-space $\tilde{y} = 0$, consist of nested two-surfaces all having the line $u = 1$ in common.

It is then possible show that the hypersurface $u = 1$ is not merely a coordinate singularity, but is actually a singularity of space-time in the sense that there does not exist a C^1 extension from region II to this surface. This can not be a curvature singularity, since the curvature tensor on it is zero. Matzner and Tipler accordingly describe it as a ‘fold singularity’. With this interpretation, the structure of the Khan–Penrose solution is thus as described in Figure 8.4.

This result raises another problem. In the absence of the second wave, the entire space-time may be described by (8.3) for $u < 0$, and by (8.4) for $u \geq 0$, for all values of v. In this case, $u = 1$ is merely a coordinate singularity. It has no physical significance and can be removed by a coordinate transformation. However, once the second wave is present,

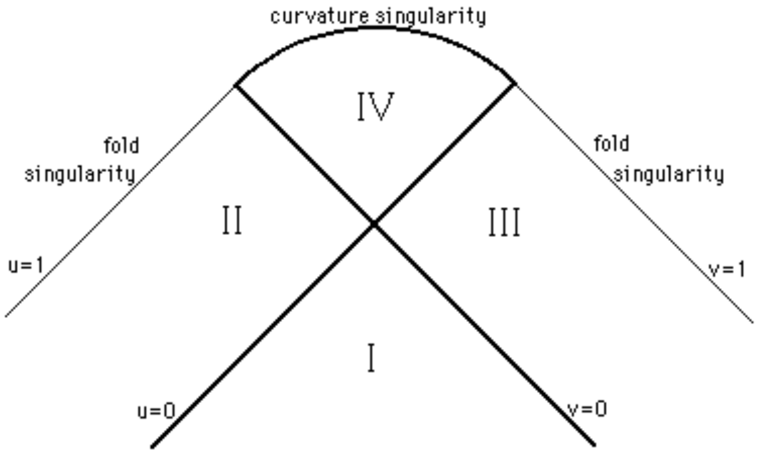

Figure 8.4 The singularity structure of the Khan–Penrose solution.

with wavefront $v = 0$, the above result states that $u = 1$ becomes a space-time singularity even for $v < 0$. This seems to violate our familiar concept of causality. The presence of the second wave seems to change the character of the prior singularity.

The resolution of this problem can be demonstrated using Figure 8.3. The family of surfaces for $v =$ constant can be continued up to $v = 0$. However, there is a curvature singularity at $u = 1$, $v = 0$. The final surface $v = 0$, which forms a bound for the prior surfaces, therefore contains a singularity on the line $u = 1$. In fact, this line $u = 1$, $v = 0$ is actually common to the entire family of surfaces for which $v \leq 0$. Thus the line $u = 1$, $v \leq 0$ must be considered as a singularity of space-time, which is identified with the point $u = 1$, $v = 0$. In the Khan–Penrose solution, the singularities in regions II and III are topological singularities that are identified with the curvature singularity in region IV by the particular choice of coordinates.

This point may be further clarified by considering a family of null geodesics that are initially parallel in region I, and enter region II. The null geodesic $x = y = 0$, $v =$ constant, starts in region I, passes through the gravitational wave and apparently ends at the $u = 1$ singularity. This may be contrasted, however, with a neighbouring geodesic which is given using the the null Minkowski coordinates $\tilde{x}^\mu$ defined by (8.5), by $\tilde{x} = \epsilon$, $\tilde{y} = 0$, $\tilde{v} =$ constant $= \tilde{v}_0$. This geodesic is given by

$$x = \frac{\epsilon}{1-u}, \qquad y = 0, \qquad v = \tilde{v}_0 + \frac{\frac{1}{2}\epsilon^2}{1-u}. \tag{8.6}$$

From this it is clear that v increases indefinitely as $u \to 1$ even for arbitrarily small values of ϵ. Thus, although the geodesic with $x = y = 0$

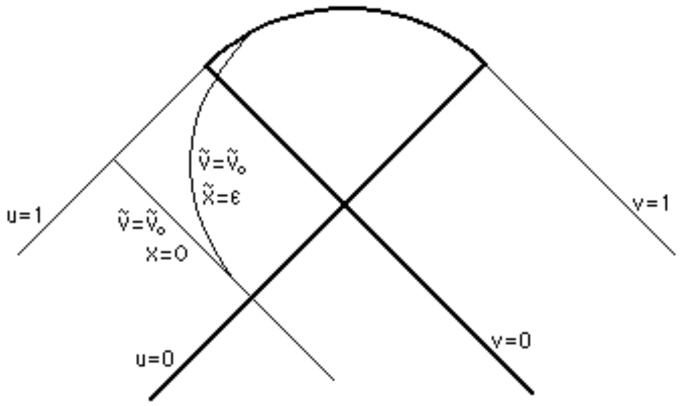

Figure 8.5 Projections of two neighbouring null geodesics onto the plane $x = 0$, $y = 0$. The geodesics are parallel and arbitrarily close in region II, which is flat.

approaches the 'fold singularity' apparently at a finite distance from the curvature singularity in region IV, an arbitrarily close geodesic that is initially parallel to it in region II diverges from it and crosses into region IV before it reaches the hypersurface $u = 1$. This geodesic subsequently terminates in the curvature singularity in region IV. The projection of these two geodesics onto the plane $x = y = 0$ is illustrated in Figure 8.5.

It has thus been argued that the singularities $u = 1$ and $v = 1$ in regions II and III are essentially extensions of the singularity $f + g = 0$ in region IV. The apparent non-causality that appears in the introduction of these singularities is a consequence of the projection of the space-time onto the plane $x = y = 0$. Such a projection does not in general preserve causal relations.

Although Figure 8.4 is very useful in formulating the colliding plane wave problem by dividing the space-time up into appropriate regions, in many ways it is misleading. In particular, it does not adequately describe the singularity structure of the solution, which appears to be non-causal. The problem arises from the attempt to represent space-time on a two-dimensional diagram. In many ways it would be preferable to attempt to picture a three-dimensional structure as in Figure 8.3. Such a picture has been drawn by Penrose, and was included in the paper of Matzner and Tipler (1984). This structure is illustrated in Figure 8.6 where the four regions are separately represented. Since curvature occurs only on the boundaries of regions I, II and III, the differently shaped boundaries of these regions have to be identified artificially. Region IV is curved and no adequate representation of it can be given.

Using the classification scheme of Ellis and Schmidt (1977), it can be seen that the singularity in region IV of the Khan–Penrose solution is

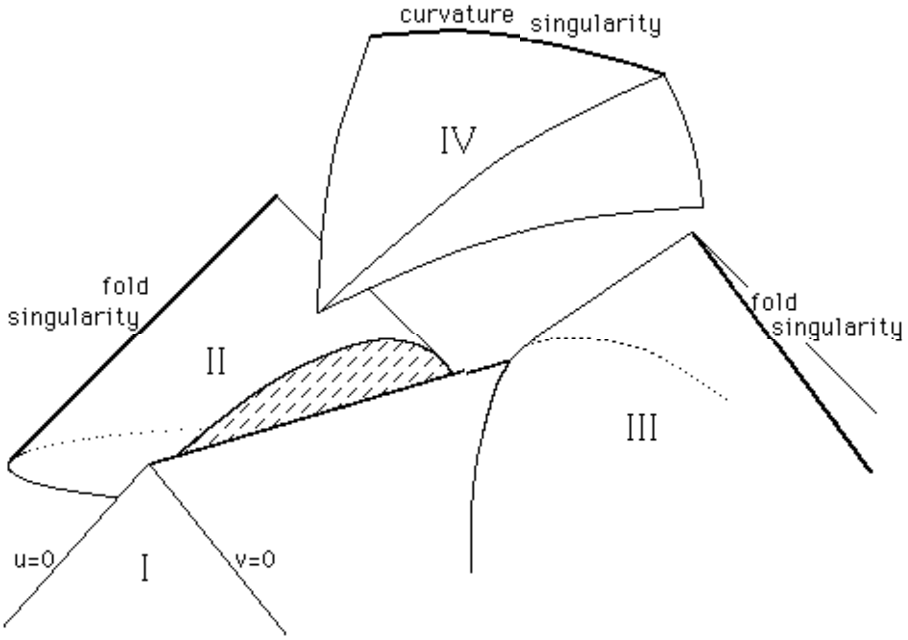

Figure 8.6 A three-dimensional picture of the Khan–Penrose solution with $y = 0$. The interiors of regions I, II and III are flat, but the boundaries have intrinsically different geometry. Points on opposite sides of the wavefronts $u = 0$ and $v = 0$ must be identified. Region IV is curved, so its representation should be considered no more than schematic.

clearly a scalar polynomial curvature singularity. Also the fold singularities in regions II and III that have a topological character, are quasiregular singularities since on them the curvature tensor is zero.

8.4 The structure of other solutions

In the previous section the singularity structure of the Khan–Penrose solution has been analysed in some detail. This has been possible because the interiors of regions I, II and III are all flat. In the general problem, however, it is only the background region I that is taken to be flat, and regions II and III, as well as IV, are curved. Consequently it is not possible in general to analyse the singularity structure in such detail.

It is reasonable to assume, however, that most colliding plane wave solutions will have the same general singularity structure as that of the Khan–Penrose solution. This in fact turns out to be the case, as will be shown here and in the following chapters. In almost all cases a curvature singularity develops in region IV on the hypersurface $f + g = 0$, although there is a large class of exceptional solutions in which the singularity is replaced by a Cauchy horizon. These exceptional solutions provide the only significant variation of the general singularity structure described above. The structure of these solutions will be described as they are derived in the following chapters.

It has also been shown that, for all colliding plane wave solutions, coordinate singularities necessarily occur in regions II and III. For vacuum solutions, in these regions at most one component, either Ψ_4 or Ψ_0, of the curvature tensor is non-zero. Scalar polynomial curvature singularities therefore cannot occur. The coordinate singularities on the hypersurfaces $f = -\frac{1}{2}$ and $g = -\frac{1}{2}$ may thus either be quasiregular singularities if the curvature tensor on these surfaces is bounded, or they must be non-scalar curvature singularities if the curvature components become unbounded.

It will now be argued that the topological 'fold' singularities of regions II and III in the Khan–Penrose solution, are also general features of all colliding plane wave solutions.

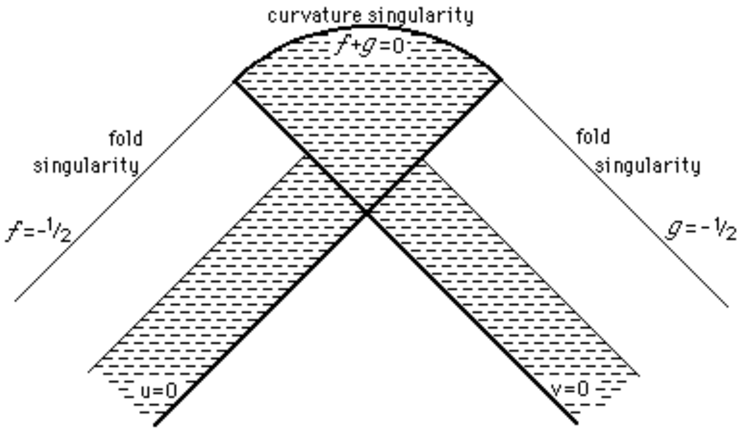

Figure 8.7 The singularity structure for colliding sandwich gravitational waves. The shaded regions have non-zero curvature.

If the approaching waves are 'sandwich' gravitational waves, then the regions behind the waves are flat. These flat regions must be described by metrics that are equivalent to those of the Khan–Penrose solution for regions II and III. After all, the impulsive waves considered by Khan and Penrose may be regarded as idealizations of such waves. In these cases similar 'fold' singularities will occur, as described in Figure 8.7.

These singularities in regions II and III have been more thoroughly investigated by Konkowski and Helliwell (1989). By concentrating on the Szekeres (1972) family of solutions, they have shown that these are quasiregular singularities in the classification scheme of Ellis and Schmidt (1977) in the cases of impulsive and sandwich waves. In the alternative case of thick gravitational waves, they are non-scalar curvature singularities. In addition, by considering scalar wave perturbations in the impulsive wave case, they have also shown that the quasiregular singularities are unstable and convert to scalar curvature singularities.

To consider further the general character of these coordinate singularities in regions II and III we may concentrate on region II. The metric in this region may be taken to be of the form (6.20), but with the metric functions U, V, W and M all depending on u only. These functions may be obtained from those in region I simply by replacing g by $\frac{1}{2}$, or v by 0. Considering the geodesics in this region, it is clear that there exist three conserved momentum components p_x, p_y and p_v besides the energy integral. These are given by

$$\begin{aligned} p_x &= -2e^{-U}(e^V \cosh W\dot{x} - \sinh W\dot{y}) \\ p_y &= -2e^{-U}(e^{-V} \cosh W\dot{y} - \sinh W\dot{x}) \\ p_v &= 2e^{-M}\dot{u} \qquad (8.7) \\ \epsilon &= 2e^{-M}\dot{u}\dot{v} - e^{-U}\left(e^V \cosh W\dot{x}^2 - 2\sinh W\dot{x}\dot{y} + e^{-V}\cosh W\dot{y}^2\right) \\ &= p_v\dot{v} - \frac{e^{-U}}{4}\left(e^{-V}\cosh Wp_x^2 + 2\sinh Wp_xp_y + e^V\cosh Wp_y^2\right) \end{aligned}$$

where ϵ may be taken to be 0 on null geodesics and 1 on time-like geodesics.

Using (8.7), we obtain the equation

$$\begin{aligned} p_v^2\frac{\mathrm{d}v}{\mathrm{d}u} = {} & \frac{e^{-(U+M)}}{2}\left(e^{-V}\cosh Wp_x^2 + 2\sinh Wp_xp_y + e^V\cosh Wp_y^2\right) \\ & + 2\epsilon\, e^{-M}. \qquad (8.8) \end{aligned}$$

Expressions for U and M in region II can be obtained from (7.2) and (7.8) in the form

$$e^{-U} = (\tfrac{1}{2} + f), \qquad e^{-M} = -\frac{f'}{\sqrt{\frac{1}{2}+f}}e^{-S_1(f)}. \qquad (8.9)$$

With these, (8.8) may be integrated to give

$$\begin{aligned} p_v^2 v = -\int\Bigg(& \frac{e^{-S_1}}{2(\frac{1}{2}+f)^{3/2}}\left(e^{-V}\cosh Wp_x^2 + 2\sinh Wp_xp_y + e^V\cosh Wp_y^2\right) \\ & + 2\epsilon\frac{e^{-S_1}}{\sqrt{\frac{1}{2}+f}}\Bigg)\mathrm{d}f. \qquad (8.10) \end{aligned}$$

It may immediately be seen from this that no time-like or null geodesics with either p_x or p_y non-zero can avoid crossing the surface $v = 0$ into region IV before reaching the surface on which $f=-\frac{1}{2}$. The only exception occurs when $p_x=p_y=0$. This clearly manifests the character of the fold singularity as described in the previous section.

In colliding plane wave problems, what is essentially a coordinate singularity for a single wave is transformed into a fold singularity prior to the collision by the presence of curvature singularities at the points $v=0$, $f=-\frac{1}{2}$, and $u=0$, $g=-\frac{1}{2}$. Generally these are associated with a space-like singularity on $f+g=0$, but as described above, there are exceptional cases in which this singularity is replaced by a horizon. However, even in these cases, it is found that there exist at least distribution valued singularities just at these points. From this, it may be argued that the presence of fold singularities in regions II and III appears to be a general feature of colliding plane wave problems.

It has thus been argued that the general structure of all colliding plane wave solutions is as illustrated in Figure 8.6 with $y=0$ with the possible exception that, for some solutions, the curvature singularity in region IV is replaced by a Cauchy horizon.

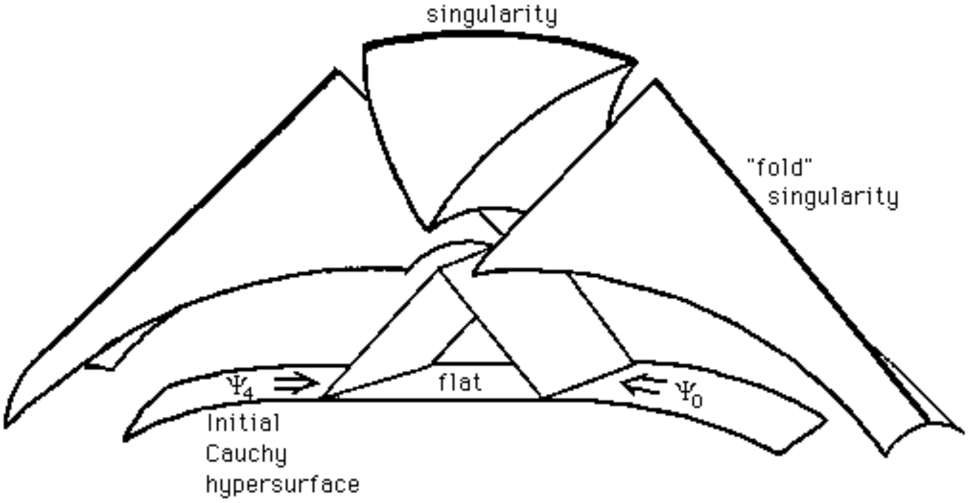

Figure 8.8 The general structure of colliding plane wave solutions related to the initial hypersurface on which initial Cauchy data is set.

It is also appropriate to attempt to relate this general singularity structure to some initial hypersurface on which the initial Cauchy data for the colliding wave problem is specified. To do this it is necessary to incorporate the properties of plane waves that have been described in Section 4.4 and illustrated in Figure 4.1. In particular, it must be noted that no global space-like hypersurface exists on which initial Cauchy data for the problem can be set. The existence of the topological 'fold' singularities in regions II and III implies bounds for the initial Cauchy hypersurface in both of the directions from which the waves emerge. This property is illustrated in Figure 8.8.

It can immediately be deduced from Figure 8.8 that, for solutions in which the curvature singularity in region IV is replaced by a Cauchy horizon, any possible extension through this horizon must be non-unique. Any extended space-time beyond this singularity will depend on extra initial data in addition to that specified on the initial Cauchy hypersurface for the colliding wave problem.

The above arguments lead to the conclusion that the usual initial data for colliding plane waves leads to a unique solution only up to the topological singularities $f=-\frac{1}{2}$ and $g=-\frac{1}{2}$ in regions II and III, and to the 'focusing' singularity $f+g=0$ in the interaction region. This focusing singularity is normally a curvature singularity, but this may be replaced by a Cauchy horizon. In this case, any future extension through the horizon is non-unique.